

Duality for relative Prymians associated to K3 double covers of del Pezzo surface of degree 2

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Abstract

Markushevich and Tikhomirov provided a construction of an irreducible symplectic V-manifold of dimension 4, the relative compactified Prym variety of a family of curves with involution, which is a Lagrangian fibration with polarization of type (1,2). We give a characterization of the dual Lagrangian fibration. We also identify the moduli space of Lagrangian fibrations of this type and show that the duality defines a rational involution on it.

Introduction

Irreducible symplectic varieties are defined as compact holomorphically symplectic Kähler varieties with trivial fundamental group, whose symplectic structure is unique up to proportionality.

By the Bogomolov decomposition Theorem [4], irreducible symplectic varieties play (together with Calabi Yau manifolds and complex tori) a central role in the classification of compact Kähler manifolds with torsion c_1 .

Very few deformation classes of irreducible symplectic varieties are known. For any positive integer n , Beauville exhibited 2 examples of dimension $2n$ in [3]: the Hilbert scheme $X^{[n]}$ parametrizing 0 dimensional subschemes of length n on a K3 surface X , and the generalized Kummer variety $K^n(T)$ of a 2-dimensional torus T , namely the locus in $X^{[n+1]}$ parametrizing the subschemes whose associated cycle sums up to 0 in X .

Besides the Beauville examples, there are only two more known constructions of irreducible symplectic varieties up to deformation equivalence, produced by O'Grady in [17] and [18], and their dimensions are respectively ten and six.

The problem of extending the very short list of known deformation classes of irreducible symplectic varieties is very hard. One can obtain a larger stock of examples if one turns back to the original setting of Fujiki (see [6]), who considered symplectic V-manifolds. A V-manifold is an algebraic variety with at worst finite quotient singularities. We will say that a V-manifold is symplectic if its

nonsingular locus is endowed with an everywhere nondegenerate holomorphic 2-form. We will say moreover that a symplectic V-manifold is irreducible if it is complete, simply connected, and if the holomorphic 2-form is unique up to \mathbb{C}^* . Such varieties should appear as factors in the generalized Bogomolov decomposition conjecture (see [9] and [13]). All the examples of symplectic V-manifolds given by Fujiki, up to deformations of complex structure, are partial resolutions of finite quotients of the products of two symplectic surfaces. Markushevich and Tikhomirov provide in [11] a new construction of an irreducible symplectic V-manifold \mathcal{P} of dimension 4, the relative compactified Prym variety of some family of curves with involution. The structure map of the relative compactified Prym variety is a Lagrangian fibration which has a (1,2)-polarized Prym surface as generic fiber. A natural question is what the dual of this (1,2)-polarized fibration is.

The irreducible symplectic varieties and V-manifolds with a Lagrangian fibration are of particular interest, as they generalize K3 surfaces with elliptic pencil on one hand, and the phase spaces of algebraically integrable systems on the other hand. The problem of constructing the dual of a Lagrangian fibration is discussed in [22], where an interesting link to the twisted Fourier-Mukai transform is uncovered.

The construction of \mathcal{P} starts from a pair of totally tangent plane quartics \overline{B}_0 and $\overline{\Delta}_0$. The first is used to construct a degree 2 del Pezzo surface X , and the second determines a K3 double cover S of X . Then the wanted family of curves is a non-complete linear system of curves on S , and \mathcal{P} is its relative compactified Prymian. Permuting the roles of \overline{B}_0 , $\overline{\Delta}_0$, we obtain another K3 surface \tilde{S} and the corresponding Prymian $\tilde{\mathcal{P}}$.

We will prove that the Lagrangian fibrations on \mathcal{P} and $\tilde{\mathcal{P}}$ are dual to each other. Moreover we will prove that not only $S \not\cong \tilde{S}$ for generic S , but also that the derived categories of \tilde{S} , S are non-equivalent and $\tilde{S}^{[2]} \not\cong S^{[2]}$. This will allow us to conclude that the associated compactified Prymians $\tilde{\mathcal{P}}$, \mathcal{P} are non-isomorphic.

In the first section we will recall the construction of Markushevich and Tikhomirov [11]. In the second section, we will give a characterisation of the dual of a (1,2)-polarized Prym surface following Barth [1]. In the third section, we will use results of Yoshikawa [24] on moduli of 2-elementary K3 surfaces which will allow us to identify the moduli space \mathfrak{P} of compactified Prymians \mathcal{P} and to conclude that, in the generic case, the "dual" K3 surfaces S , \tilde{S} are derived equivalent if and only if they are isomorphic. And in the forth section we will finally prove that the dual Prymian $\tilde{\mathcal{P}}$ is generically non-isomorphic to \mathcal{P} , so that the duality is indeed a non-trivial involution on \mathfrak{P} .

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1 Construction of \mathcal{P}

In this section we follow [11]. We will start by the construction of the family of genus 3 curves with involution, whose relative compactified Prymian \mathcal{P} is the irreducible symplectic V-manifold discussed in the introduction.

Let \overline{B}_0 be a smooth quartic curve in \mathbb{P}^2 . Let $\mu : X \rightarrow \mathbb{P}^2$ be the double cover branched in \overline{B}_0 . Then X is a Del Pezzo surface of degree 2. Let $\overline{\Delta}_0$ be a smooth quartic curve in \mathbb{P}^2 tangent to \overline{B}_0 at eight points lying on a conic. We denote $B_0 = \mu^{-1}(\overline{\Delta}_0)$. We have $\mu^{-1}(\overline{\Delta}_0) = \Delta_0 + i(\Delta_0)$, where Δ_0 is a smooth curve from the linear system $|-2K_X|$ (see for instance Lemma 5.14 of [10]). Finally, let $\rho : S \rightarrow X$ be the double cover branched in Δ_0 , $\Delta = \rho^{-1}(\Delta_0)$. Note that if we take a similar double cover branched in $i(\Delta_0)$, $\rho' : S' \rightarrow X$, we get a surface S' isomorphic to S (indeed, $i \circ \rho'$ and ρ are two double covers branched in the same curve in X). Denote τ the involution of S induced by ρ . We have the following diagram:

$$\begin{array}{ccccc}
 & & B_0 & \xlongequal{\quad} & \overline{B}_0 \\
 & & \downarrow & & \downarrow \\
 \tau \curvearrowright S & \xrightarrow[\rho]{2:1} & X & \xrightarrow[\mu]{2:1} & \mathbb{P}^2 \\
 & & \uparrow & & \\
 & & \Delta & \xlongequal{\quad} & \Delta_0
 \end{array} \tag{1}.$$

We also allow the case where \overline{B}_0 is a quartic and $\overline{\Delta}_0$ is equal to a double conic $2Q$ such that $\Delta_0 = \mu^{-1}(Q)$ is a smooth curve. In this case $i(\Delta_0) = \Delta_0$ and Δ_0 is in $|-2K_X|$. We will have some additional conditions for matching with [11]. Define also $H = \rho^*(-K_X)$.

The involution τ of the double cover $\rho : S \rightarrow X$ is H -linear and induces an involution on $|H| \simeq \mathbb{P}^3$, whose fixed locus consists of two components: a point and a plane. The plane parametrizes the curves of the form $\rho^{-1}\mu^{-1}(t)$, where t is a line in \mathbb{P}^2 . Thus this plane is parametrized by the dual of \mathbb{P}^2 , denoted $\mathbb{P}^{2\vee}$. Let $\epsilon : \mathcal{C} \rightarrow \mathbb{P}^{2\vee}$ be the linear subsystem of τ -invariant curves parametrized by $\mathbb{P}^{2\vee}$. The properties of this linear subsystem must be as in [11].

Then, we need the following conditions. The plane quartic \overline{B}_0 must have exactly 28 bitangent lines m_1, \dots, m_{28} . The curve $\mu^{-1}(m_i)$ is the union of two rational curves $l_i \cup l'_i$ meeting in 2 points. The 56 curves l_i, l'_i are all the lines on X , that is, curves of degree 1 with respect to $-K_X$. Further, the curves $C_i = \rho^{-1}(l_i)$, $C'_i = \rho^{-1}(l'_i)$ are plane conics on S with respect to the injection $S \hookrightarrow \mathbb{P}^3$ defined by $|H|$. The conics C_i, C'_i must be irreducible and meeting in exactly 4 distinct points.

Consequently we require the following conditions for the couple $(\overline{B}_0, \overline{\Delta}_0)$.

- The quartic \overline{B}_0 must not have tangent line with multiplicity 4 in a point. In this case \overline{B}_0 has exactly 28 bitangent lines.

- A bitangent line of $\overline{B_0}$ tangent at $\overline{B_0}$ in a point p must not be tangent at $\overline{\Delta_0}$ in this same point p . In this case, the conics C_i, C'_i are not tangent, so meet in exactly 4 distinct points.
- The quartics $\overline{B_0}$ and $\overline{\Delta_0}$ must not have common bitangent line. In this case the conics C_i and C'_i are irreducible. Moreover S contains no lines.

We will denote \mathfrak{L} the set of such pairs $(\overline{B_0}, \overline{\Delta_0})$. In this section $(\overline{B_0}, \overline{\Delta_0})$ will be in \mathfrak{L} .

Let $\mathcal{M} = M_S^{H,s}(0, H, -2)$ be the moduli space of semistable sheaves \mathcal{F} on S with respect to the ample class H with Mukai vector $v(\mathcal{F}) = (0, H, -2)$. We define an involution on \mathcal{M} by

$$\sigma : \mathcal{M} \rightarrow \mathcal{M}, [\mathcal{L}] \mapsto [\mathcal{E}xt_{\mathcal{O}_S}^1(\mathcal{L}, \mathcal{O}_S(-H))],$$

and we set $\kappa = \tau^* \circ \sigma$. One can prove that κ is a regular involution on \mathcal{M} and that its fixed locus has one 4-dimensional irreducible component plus 64 isolated points.

Definition 1.1. *We define the compactified Prymian \mathcal{P} as the 4-dimensional component of $\text{Fix}(\kappa)$.*

Theorem 1.2. *The variety \mathcal{P} is an irreducible symplectic V -manifold of dimension 4 with only 28 singular points analytically equivalent to $(\mathbb{C}^4 / \{\pm 1\}, 0)$.*

Proof. See Theorem 3.4, Proposition 5.4 and Corollary 5.7 of [11]. \square

Now, we will introduce the Lagrangian fibration. We consider the linear subsystem $\epsilon : \mathcal{C} \rightarrow \mathbb{P}^{2\vee}$. If $t \in \mathbb{P}^{2\vee}$ is not tangent to $\overline{B_0}$ neither to $\overline{\Delta_0}$, which is the generic case, then $C_t = \epsilon^{-1}(t) = \rho^{-1}\mu^{-1}(t)$ is a smooth genus-3 curve, and $E_t = C_t/\tau$ is elliptic. The double cover $\rho_t = \rho|_{C_t} : C_t \rightarrow E_t$ is branched at 4 points of the intersection $\Delta_0 \cap E_t$ and the double cover $\mu_t = \mu|_{E_t} : E_t \rightarrow t \simeq \mathbb{P}^1$ is branched at 4 points of the intersection $\overline{B_0} \cap t$. We denote also $\tau_t = \tau|_{C_t}$. Thus, we have the tower of double covers:

$$C_t \xrightarrow{2:1} E_t \xrightarrow{2:1} \mathbb{P}^1.$$

The following Lemma introduces the (1,2)-polarized Prym surfaces:

Lemma 1.3. *For a generic line $t \in \mathbb{P}^2$, $\ker(\text{id} + \tau_t)$ has only one connected component in $J(C_t)$, and the restriction of the principal polarization from $J(C_t)$ to the abelian variety $\text{Prym}(C_t, \tau_t) = \ker(\text{id} + \tau_t)$ is a polarization of type (1, 2).*

Proof. See Lemma 3.2. in [11]. \square

Theorem 1.4. *Identifying, as above, the 2-dimensional linear subsystem of τ -invariant curves in $|H|$ with $\mathbb{P}^{2\vee}$, let $f_{\mathcal{P}} : \mathcal{P} \rightarrow \mathbb{P}^{2\vee}$ the map sending each sheaf to its support. Then $f_{\mathcal{P}}$ is a Lagrangian fibration on \mathcal{P} and the generic fiber $f_{\mathcal{P}}^{-1}(t)$ is the (1,2)-polarized Prym surface $\text{Prym}(C_t, \tau_t)$.*

Proof. See Theorem 3.4 of [11]. \square

In fact, \mathcal{P} is bimeromorphic to a partial resolution of a quotient of $S^{[2]}$. Consider Beauville's involution (see Section 6 of [2]):

$$\iota_0 : S^{[2]} \rightarrow S^{[2]}, \xi \mapsto \xi' = (\langle \xi \rangle \cap S) - \xi.$$

Here S is taken in its embedding as a quartic surface in \mathbb{P}^3 , given by the linear system $|H|$, $\langle \xi \rangle$ stands for the line in \mathbb{P}^3 spanned by ξ , and ξ' is the residual intersection of $\langle \xi \rangle$ with S . By [2], this involution is regular whenever S contains no lines, which is true in our case. Further, τ induces on $S^{[2]}$ an involution which we will denote by the same symbol. As τ on S is the restriction of a linear involution on \mathbb{P}^3 , ι_0 commutes with τ , and the composition $\iota = \iota_0 \circ \tau$ is also an involution.

Proposition 1.5. *The fixed locus of ι is the union of a K3 surface $\Sigma \subset S^{[2]}$ and 28 isolated points.*

Proof. See Lemma 5.3 of [11]. In fact, as follows the recent work of Mongardi from [12], the fixed locus of any symplectic involution on an irreducible symplectic variety deformation equivalent to Hilbert square of a K3 surface is as in the statement of the proposition. \square

We denote $M = S^{[2]}/\iota$ and $\overline{\Sigma}$ the image of Σ in M . We also denote by M' the partial resolution of singularities of M obtained by blowing up $\overline{\Sigma}$, and by $\overline{\Sigma}'$ the exceptional divisor of the blowup.

Theorem 1.6. *The variety M' is an irreducible symplectic V-manifold whose singularities are 28 points of analytic type $(\mathbb{C}^4/\{\pm 1\}, 0)$. Moreover there is a Mukai flop between M' and \mathcal{P} , which is an isomorphism between $M' \setminus \Pi'$ and $\mathcal{P} \setminus \Pi$, where Π' and Π are Lagrangian subvarieties isomorphic to \mathbb{P}^2 .*

Proof. See Corollary 5.7 of [11]. \square

2 The (1,2)-polarized Prym surfaces

In this section \overline{B}_0 and $\overline{\Delta}_0$ are smooth quartics tangent to each other at eight points lying on a conic, (we will denote U the set of such pairs). Moreover, we assume that the pairs $(\overline{B}_0, \overline{\Delta}_0)$ and $(\overline{\Delta}_0, \overline{B}_0)$ are in \mathfrak{L} . We can permute the roles of $\overline{\Delta}_0$ and \overline{B}_0 in the above construction. Namely, consider the double cover $\tilde{\mu} : \tilde{X} \rightarrow \mathbb{P}^2$ branched in $\overline{\Delta}_0$. Denote $\tilde{\Delta}_0 = \tilde{\mu}^{-1}(\overline{\Delta}_0)$, and let \tilde{B}_0 and \tilde{B}'_0 be the two curves mapped to \overline{B}_0 by $\tilde{\mu}$. We denote by \tilde{i} the involution of \tilde{X} induced by $\tilde{\mu}$ which exchanges \tilde{B}_0 and \tilde{B}'_0 . Consider the double cover $\tilde{\rho} : \tilde{S} \rightarrow \tilde{X}$ branched in \tilde{B}_0 and set $\tilde{B} = \tilde{\rho}^{-1}(\tilde{B}_0)$. And, denote by $\tilde{\tau}$ the involution of \tilde{S} induced by $\tilde{\rho}$. We have the diagram

$$\begin{array}{ccccc}
& & \widetilde{\Delta}_0 & \xlongequal{\quad} & \overline{\Delta}_0 \\
& & \downarrow & & \downarrow \\
\tilde{\tau} \curvearrowright \widetilde{S} & \xrightarrow[\tilde{\rho}]{2:1} & \widetilde{X} & \xrightarrow[\tilde{\mu}]{2:1} & \mathbb{P}^2 \\
\uparrow \widetilde{B} & & \uparrow \widetilde{B}_0 & & \\
& & \widetilde{B} & \xlongequal{\quad} & \widetilde{B}_0
\end{array}$$

similar to (1).

Like in Section 1, for a generic line in \mathbb{P}^2 , we denote $\widetilde{E}_t = \widetilde{\mu}^{-1}(t)$, $\widetilde{C}_t = \widetilde{\rho}^{-1}(\widetilde{E}_t)$, $\widetilde{\mu}_t = \widetilde{\mu}|_{\widetilde{E}_t}$, $\widetilde{\rho}_t = \widetilde{\rho}|_{\widetilde{C}_t}$ and $\widetilde{\tau}_t = \widetilde{\tau}|_{\widetilde{C}_t}$. The generic curves \widetilde{E}_t are elliptic and the curves \widetilde{C}_t are of genus 3. Thus, we have the tower of double covers:

$$\widetilde{C}_t \xrightarrow{2:1} \widetilde{E}_t \xrightarrow{2:1} \mathbb{P}^1.$$

By Lemma 1.3, $\text{Prym}(\widetilde{C}_t, \widetilde{\tau}_t)$ is also a (1,2)-polarized Prym surface.

We denote by $\text{Prym}(C_t, \tau_t)^\vee$ the dual of the polarized abelian variety $\text{Prym}(C_t, \tau_t)$. The answer to one of our questions is given by the following proposition.

Proposition 2.1. *For a generic line $t \in \mathbb{P}^2$, we have the isomorphism:*

$$\text{Prym}(C_t, \tau_t)^\vee \simeq \text{Prym}(\widetilde{C}_t, \widetilde{\tau}_t).$$

Proof.

- Step 1: *The curve bigonally related to C_t*

Starting with the tower $C_t \rightarrow E_t \rightarrow \mathbb{P}^1$, we will construct a curve C_t^\vee whose points correspond to the different ways to lift the pairs $\mu_t^{-1}(p)$, for $p \in \mathbb{P}^1$, to a pair in C_t , i.e.,

$$C_t^\vee = \left\{ p + q \in \text{Div}^{(2)}(C_t) \mid [\rho_t(p) + \rho_t(q)] = [\mu_t^* \mathcal{O}_{\mathbb{P}^1}(1)] \right\}.$$

We denote by τ_t^\vee the involution $C_t^\vee \rightarrow C_t^\vee$ sending a lift to its complement. We have $\tau_t^\vee = \tau_t^*|_{\text{Div}^{(2)}(C_t)}$. Let $E_t^\vee = C_t^\vee / \tau_t^\vee$. We also define the map $\mu_t^\vee : E_t^\vee \rightarrow \mathbb{P}^1$ which sends a lift of $\mu_t^{-1}(p)$ ($p \in \mathbb{P}^1$) to p .

For a better understanding we draw a diagram. Let α be a generic point in \mathbb{P}^1 (a point which is not a branch point of μ_t nor the image of a branch point of ρ_t), β_i , $i = 1$ or 2 its preimages under μ_t and $\gamma_{i,j}$, $(i, j) \in \{1, 2\}^2$, the preimages of the β_i under ρ_t , as shown in the diagram:

$$\begin{array}{ccccc}
C_t & \xrightarrow{\rho_t} & E_t & \xrightarrow{\mu_t} & \mathbb{P}^1 \\
\gamma_{1,1} & \nearrow & \beta_1 & \longrightarrow & \alpha \\
\gamma_{1,2} & \nearrow & & & \\
\gamma_{2,1} & \nearrow & \beta_2 & & \\
\gamma_{2,2} & \nearrow & & &
\end{array}$$

This gives the following diagram for points in C_t^\vee and E_t^\vee :

$$\begin{array}{ccccc}
C_t^\vee & \xrightarrow{\rho_t^\vee} & E_t^\vee & \xrightarrow{\mu_t^\vee} & \mathbb{P}^1 \\
\gamma_{1,1} + \gamma_{2,1} & \longrightarrow & \overline{\gamma_{1,1} + \gamma_{2,1}} = \overline{\gamma_{1,2} + \gamma_{2,2}} & \longrightarrow & \alpha. \\
\gamma_{1,2} + \gamma_{2,2} & \longrightarrow & & & \\
\gamma_{1,1} + \gamma_{2,2} & \longrightarrow & \overline{\gamma_{1,1} + \gamma_{2,2}} = \overline{\gamma_{1,2} + \gamma_{2,1}} & \longrightarrow & \\
\gamma_{1,2} + \gamma_{2,1} & \longrightarrow & & &
\end{array}$$

Like ρ_t , μ_t the maps ρ_t^\vee , μ_t^\vee are double covers:

$$C_t^\vee \xrightarrow{2:1} E_t^\vee \xrightarrow{2:1} \mathbb{P}^1. \quad (*)$$

Barth [1] calls this way to obtain C_t^\vee Pantazis's bigonal construction (see [20], p. 304).

Proposition 2.2. *The abelian varieties $\text{Prym}(C_t, \tau_t)$ and $\text{Prym}(C_t^\vee, \tau_t^\vee)$ are dual to each other in such a way that C_t^\vee (resp. C_t) embeds in $\text{Prym}(C_t, \tau_t)$ (resp. $\text{Prym}(C_t^\vee, \tau_t^\vee)$) as a theta-divisor of a polarisation of type $(1, 2)$.*

Proof. See [20] Proposition 1 Section 3 page 307. \square

Now, we will show that $\text{Prym}(C_t^\vee, \tau_t^\vee)$ and $\text{Prym}(\widetilde{C}_t, \widetilde{\tau}_t)$ are isomorphic. To this end, we will look what happens when α is a branch point.

- Step 2: *The ramification of the double covers of the diagram (*)*

We will denote by $(a_i)_{1 \leq i \leq 4}$ the branch points of μ_t , $(b_i)_{1 \leq i \leq 4}$ their preimages in E_t , $(e_i)_{1 \leq i \leq 4}$ the branch points of ρ_t , $(p_i)_{1 \leq i \leq 4}$ their images in \mathbb{P}^1 , $(e'_i)_{1 \leq i \leq 4}$ the other preimages of the p_i in E_t , $(c_i)_{1 \leq i \leq 4}$ the preimages of the $(e_i)_{1 \leq i \leq 4}$ in C_t , $(b_{i,j})_{1 \leq i \leq 4, 1 \leq j \leq 2}$ the preimages of the $(b_i)_{1 \leq i \leq 4}$ in C_t and $(c'_{i,j})_{1 \leq i \leq 4, 1 \leq j \leq 2}$ the preimages of the $(e'_i)_{1 \leq i \leq 4}$ in C_t , as in the following diagram:

$$\begin{array}{ccccc}
C_t & \xrightarrow{\rho_t} & E_t & \xrightarrow{\mu_t} & \mathbb{P}^1 \\
b_{i,j} & \longrightarrow & b_i & \longrightarrow & a_i \\
c_i & \longrightarrow & e_i & \longrightarrow & p_i. \\
c'_{i,j} & \longrightarrow & e'_i & \nearrow &
\end{array}$$

We have $\tau_t(b_{i,1}) = b_{i,2}$, $\tau_t(c'_{i,1}) = c'_{i,2}$ for $\mu_t^{-1}(a_i) = \{b_i\}$ and $\rho_t^{-1}(\mu_t^{-1}(a_i)) = \{b_{i,1}, b_{i,2}\}$, for all $i \in [1, 4]$. So we see that the ramification points of ρ_t^\vee are the pairs $b_{i,1} + b_{i,2}$ $1 \leq i \leq 4$. We have also $\mu_t^{-1}(p_i) = \{e_i, e'_i\}$ and $\rho_t^{-1}(\mu_t^{-1}(p_i)) = \{c_i, c'_{i,1}, c'_{i,2}\}$. The involution τ_t^\vee exchanges $c_i + c'_{i,1}$ for $c_i + c'_{i,2}$. And the classes c'_i of the pairs $c_i + c'_{i,1}$ and $c_i + c'_{i,2}$ in E_t^\vee , $1 \leq i \leq 4$, are the ramification points of μ_t^\vee . We show this in the diagram:

$$\begin{array}{ccccc}
C_t^\vee & \xrightarrow{\rho_t^\vee} & E_t^\vee & \xrightarrow{\mu_t^\vee} & \mathbb{P}^1 \\
c_i + c'_{i,j} & \longrightarrow & \overline{c_i + c'_{i,1}} = \overline{c_i + c'_{i,2}} & \longrightarrow & p_i \\
b_{i,1} + b_{i,2} & \longrightarrow & \overline{b_{i,1} + b_{i,2}} & \longrightarrow & a_i. \\
2b_{i,j} & \longrightarrow & \overline{2b_{i,1}} = \overline{2b_{i,2}} & \searrow &
\end{array}$$

• Step 3: *Conclusion*

We see that the maps μ_t^\vee and $\tilde{\mu}_t$ have the same branch points in \mathbb{P}^1 by Step 2. This gives an isomorphism φ_t between E_t^\vee and \tilde{E}_t . Now, we want that φ_t sends the branch points of ρ_t^\vee to the branch points of $\tilde{\rho}_t$ to build an isomorphism between \tilde{C}_t and C_t^\vee . To do this, we can send E_t^\vee to \tilde{X} by φ_t . Then, by Step 2, we see that the branch points of ρ_t^\vee are necessarily on \tilde{B}_0 or \tilde{B}'_0 . Indeed, their images by μ_t^\vee in \mathbb{P}^1 are the branch points of μ_t , which are in $\overline{B_0}$. If the branch points of ρ_t^\vee are on \tilde{B}'_0 , we just need to compose φ_t with \tilde{i} (the involution on \tilde{X} we have defined in the very beginning) to obtain an isomorphism between E_t^\vee and \tilde{E}_t which sends the branch points of ρ_t^\vee to the branch points of $\tilde{\rho}_t$. Denote this isomorphism by φ_t . Then we obtain the commutative diagram

$$\begin{array}{ccccc}
\tilde{C}_t & \xrightarrow{\tilde{\rho}_t} & \tilde{E}_t & \xrightarrow{\tilde{\mu}_t} & \mathbb{P}^1, \\
\Big\downarrow & & \Big\downarrow \varphi_t & \nearrow \mu_t^\vee & \\
C_t^\vee & \xrightarrow{\rho_t^\vee} & E_t^\vee & &
\end{array}$$

which implies $\text{Prym}(C_t^\vee, \tau_t^\vee) \simeq \text{Prym}(\tilde{C}_t, \tilde{\tau}_t)$.

□

3 Relation between S and \tilde{S}

It is a natural question to know whether the two K3 surfaces S, \tilde{S} are isomorphic or not. We are going to prove that the answer is no for generic S . The meaning of "generic" will be clear from what follows.

3.1 Definition of $\mathfrak{M}_{r,a,\delta}$

Before giving the definition of $\mathfrak{M}_{r,a,\delta}$, we need some notions and some notation about lattices. For a lattice M , we will denote its rank $r(M)$. The signature of M will be denoted by $\text{sign } M = (b^+(M), b^-(M))$. A lattice M is *Lorentzian* if $\text{sign } M = (1, r(M) - 1)$. We will denote by M^\vee the dual of M and by $A_M = M^\vee/M$ the discriminant group. An even lattice M is *2-elementary* if there is an integer a with $A_M \simeq (\mathbb{Z}/2\mathbb{Z})^a$; then we set $a(M) = \dim_{\mathbb{Z}/2\mathbb{Z}} A_M$.

We also define $\delta(M) = 0$ if $x^2 \in \mathbb{Z} \ \forall x \in M^\vee$, otherwise $\delta(M) = 1$. The triple $(\text{sign}(M), a(M), \delta(M))$ determines the isometry class of an indefinite even 2-elementary lattice M by Theorem 3.6.2 of [14].

Let S be a K3 surface equipped with an antisymplectic involution $\tau : S \rightarrow S$. We denote $P = \text{Pic}(S)^\tau$. Then P is a primitive 2-elementary Lorentzian sublattice of $H^2(S, \mathbb{Z})$ endowed with the cup product (see for instance Lemma 1.3 of [23]). Let (r, a, δ) be a triple of integers. A couple (S, τ) is called a *2-elementary K3 surface of type (r, a, δ)* if $(r(P), a(P), \delta(P)) = (r, a, \delta)$. We denote by $\mathfrak{M}_{r,a,\delta}$ the moduli space of isomorphism classes of 2-elementary K3 surfaces of type (r, a, δ) .

For a K3 surface S , $H^2(S, \mathbb{Z})$ endowed with the cup-product pairing is isometric to the K3 lattice $L = E_8(-1)^2 \oplus U^3$. An isometry of lattices $\alpha : H^2(S, \mathbb{Z}) \cong L$ is called a *marking* of S . The pair (S, α) is called a *marked K3 surface*. Let $M \subset L$ be a primitive 2-elementary Lorentzian sublattice. Let I_M be the involution on $M \oplus M^\perp$ defined by

$$I_M(x, y) = (x, -y), \quad (x, y) \in M \oplus M^\perp.$$

Then I_M extends uniquely to an involution on L by Corollary 1.5.2 of [14]. A K3 surface equipped with an anti-symplectic holomorphic involution $\tau : S \rightarrow S$ is called a *2-elementary K3 surface of type M* if there exists a marking α of S satisfying

$$\tau^* = \alpha^{-1} \circ I_M \circ \alpha.$$

Such a marking will be called a *M -marking* of (S, τ) . We note that $\alpha((\text{Pic}(S))^\tau) = M$. Now we will show that a 2-elementary K3 surface of type (r, a, δ) and a 2-elementary K3 surface of type M where $(r(M), a(M), \delta(M)) = (r, a, \delta)$, are equivalent notions.

Lemma 3.1. *Let $\varphi : N_1 \simeq N_2$ be an isometry between two 2-elementary sublattices of L . We assume that $\text{sign } N_1 = \text{sign } N_2 = (2, x)$ where x is an integer, then we can extend φ to an isometry of L .*

Proof. We will use Corollary 1.5.2 of [14]. We start by showing an isometry between N_1^\perp and N_2^\perp . We have $\text{sign}(N_1^\perp) = \text{sign}(N_2^\perp) = (1, 19 - x)$ (because $\text{sign}(L) = (3, 19)$). Since L is unimodular, we have an isomorphism $\gamma_{N_1} : A_{N_1} \rightarrow A_{N_1^\perp}$ with $q_{N_1^\perp} \circ \gamma_{N_1} = -q_{N_1}$ and an isomorphism $\gamma_{N_2} : A_{N_2} \rightarrow A_{N_2^\perp}$ with $q_{N_2^\perp} \circ \gamma_{N_2} = -q_{N_2}$ (see [14] section 1). This implies $(a(N_1^\perp), \delta(N_1^\perp)) = (a(N_2^\perp), \delta(N_2^\perp))$. Then, by Theorem 3.6.2 of [14] there is an isometry $\psi : N_1^\perp \rightarrow N_2^\perp$.

One the other hand, φ (resp. ψ) induces an isometry $\overline{\varphi} : A_{N_1} \rightarrow A_{N_2}$ (resp. $\overline{\psi} : A_{N_1^\perp} \rightarrow A_{N_2^\perp}$). Now we take the following composition:

$$\gamma_{N_2} \circ \overline{\varphi} \circ \gamma_{N_1}^{-1} \circ \overline{\psi}^{-1},$$

which is an auto-isometry of $A_{N_2^\perp}$. Since N_2^\perp is a 2-elementary Lorentzian sublattice, Theorem 3.6.3 of [14] gives us an isometry $\chi \in \mathcal{O}(N_2^\perp)$ with $\overline{\chi} =$

$\gamma_{N_2} \circ \overline{\varphi} \circ \gamma_{N_1}^{-1} \circ \overline{\psi^{-1}}$. Hence

$$\gamma_{N_2} \circ \overline{\varphi} = \overline{\chi \circ \psi} \circ \gamma_{N_1}.$$

By Corollary 1.5.2 of [14], φ extends to an isometry of L . \square

Remark: The same result holds if $\text{sign } N_i = (1, x)$, as we are going to see in the proof of the next proposition.

Proposition 3.2. *A K3 surface is a 2-elementary K3 surface of type (r, a, δ) if and only if it is a 2-elementary K3 surface of type M for some primitive 2-elementary Lorentzian sublattice M with $(r(M), a(M), \delta(M)) = (r, a, \delta)$.*

Proof. It is obvious that a 2-elementary K3 surface (S, τ) of type M with $(r(M), a(M), \delta(M)) = (r, a, \delta)$ belongs to $\mathfrak{M}_{r,a,\delta}$. So, we will show the other implication. Let $M \subset L$ with $(r(M), a(M), \delta(M)) = (r, a, \delta)$ and let $(S, \tau) \in \mathfrak{M}_{r,a,\delta}$, we will find a M -marking of S . Let $P = \text{Pic}(S)^\tau$, we can assume that P is a sublattice of L , it suffices to take its image by some marking of S . We have $(r(P), a(P), \delta(P)) = (r, a, \delta)$, then by Theorem 3.6.2 of [14], we have an isometry $\psi : P \rightarrow M$. Moreover we have $\text{sign}(P^\perp) = \text{sign}(M^\perp)$ and since L is unimodular we have $(a(P^\perp), \delta(P^\perp)) = (a(M^\perp), \delta(M^\perp))$. Once more, it follows by Theorem 3.6.2 of [14] that there is an isometry $\varphi : P^\perp \rightarrow M^\perp$. By Lemma 3.1 this isometry extends to an isometry of L . We denote $\tilde{\varphi}$ this isometry, we have $\tau^* = \tilde{\varphi}^{-1} \circ I_M \circ \tilde{\varphi}$. Indeed, let $x \in H^2(S, \mathbb{Z})$, $x = \frac{p+t}{2}$ where $p \in P$ and $t \in P^\perp$, hence,

$$\tilde{\varphi}^{-1} \circ I_M \circ \tilde{\varphi} \left(\frac{p+t}{2} \right) = \tilde{\varphi}^{-1} \circ I_M \left(\frac{\tilde{\varphi}(p) + \tilde{\varphi}(t)}{2} \right) = \tilde{\varphi}^{-1} \left(\frac{\tilde{\varphi}(p) - \tilde{\varphi}(t)}{2} \right) = \frac{p-t}{2}.$$

\square

The moduli space of 2-elementary K3 surface was introduced by Nikulin in [15], see also [16] and [23] Section 1 for more details.

3.2 A Torelli Theorem for 2-elementary K3 surfaces

For a better understanding of this moduli space we will give a kind of Torelli theorem for it (see [23] and [24] for more details). We need some more notation.

Let (S, α) be a marked K3 surface. Recall the definition of the period map for marked K3 surfaces: the period of (S, α) is defined to be

$$\pi(S, \alpha) := [\alpha(\eta)] \in \mathbb{P}(L \otimes \mathbb{C}), \quad \eta \in H^0(S, \omega_S) \setminus \{0\}.$$

Let Λ be a lattice of signature $(2, n)$. We define

$$\Omega_\Lambda := \{[x] \in \mathbb{P}(\Lambda \otimes \mathbb{C}); \langle x, x \rangle = 0, \langle x, \bar{x} \rangle > 0\}.$$

Denote $\Delta_\Lambda := \{x \in \Lambda; \langle x, x \rangle = -2\}$.

For $\lambda \in \Lambda \otimes \mathbb{R}$, set $H_\lambda := \{[x] \in \Omega_\Lambda; \langle x, \lambda \rangle = 0\}$. We define the discriminant locus of Ω_Λ by

$$\mathcal{D}_\Lambda := \sum_{d \in \Delta_\Lambda / \pm 1} H_d.$$

Assume that Λ is a primitive 2-elementary sublattice of L , with Λ^\perp Lorentzian. Then we set

$$\begin{aligned}\Gamma(\Lambda) &:= \{g \in \mathcal{O}(L), I_{\Lambda^\perp} g = g I_{\Lambda^\perp}\}, \\ \Gamma_\Lambda &:= \{g|_\Lambda \in \mathcal{O}(\Lambda); g \in \Gamma(\Lambda)\}, \\ \Omega_\Lambda^\circ &:= \Omega_\Lambda \setminus \mathcal{D}_\Lambda, \quad \mathcal{M}_\Lambda^\circ := \Omega_\Lambda^\circ / \Gamma_\Lambda.\end{aligned}$$

The following theorem, due to Yoshikawa ([23] Theorem 1.8) can be thought of as Torelli Theorem for 2-elementary K3 surface:

Theorem 3.3. *Via the period map, the analytic space $\mathcal{M}_{M^\perp}^\circ$ is a coarse moduli space of 2-elementary K3 surfaces of type M .*

Proof. The proof uses the classical Torelli Theorem for K3 surface (see [21] and [5]) and results of Nikulin [15]. \square

Next, Yoshikawa improves this result in [24], proving the following proposition (Proposition 11.2 in [24]).

Proposition 3.4. *The following equality holds:*

$$\Gamma_{M^\perp} = \mathcal{O}(M^\perp).$$

Proof. The proof uses Theorem 3.6.3 and Corollary 1.5.2 of [14], the idea is the same as in the proof of Lemma 3.1 and Proposition 3.2. \square

We thus obtain the following result. Let $M \subset L$ be a primitive 2-elementary Lorentzian sublattice with $(r(M), a(M), \delta(M)) = (r, a, \delta)$. Define the map:

$$\begin{aligned}\mathfrak{M}_{r,a,\delta} &\rightarrow \Omega_{M^\perp}^\circ / \mathcal{O}(M^\perp) \\ \overline{\omega}_{r,a,\delta} : (S, \tau) &\mapsto \mathcal{O}(M^\perp) \cdot \pi(S, \alpha),\end{aligned}$$

where α is a M -marking of S .

Corollary 3.5. *The map $\overline{\omega}_{r,a,\delta}$ is an isomorphism.*

Proof. See [24] page 8. \square

We define

$$\mathfrak{D}_{r,a,\delta} = \overline{\omega}_{r,a,\delta}^{-1} \left(\left\{ \mathcal{O}(M^\perp) \cdot \eta \in \Omega_{M^\perp}^\circ / \mathcal{O}(M^\perp) \mid \langle \eta, x \rangle \neq 0, \forall x \in M^\perp \setminus \{0\} \right\} \right).$$

It is the open subset of $\mathfrak{M}_{r,a,\delta}$, the one which explains the meaning of "generic". It has the following important property.

Proposition 3.6. *Let $M \subset L$ be a 2-elementary Lorentzian sublattice with $(r(M), a(M), \delta(M)) = (r, a, \delta)$. If $(S, \tau) \in \mathfrak{D}_{r,a,\delta}$, then a M -marking of (S, τ) induces an isometry between $\text{Pic } S$ and M . In particular, all elements of $\text{Pic } S$ are invariant by τ^* .*

Proof. Indeed, let $x \in \text{Pic } S$ then $\langle x, \eta \rangle = 0$, where $\eta \in H^0(S, \omega_S) \setminus \{0\}$. Let α be a M -marking of S , then we can write $\alpha(x) = \frac{a+b}{2}$, where $a \in M$ and $b \in M^\perp$. Since $\langle a, \alpha(\eta) \rangle = 0$, we have $\langle b, \alpha(\eta) \rangle = 0$, so by hypothesis $b = 0$. \square

Corollary 3.7. *Let (S, τ) and (S', τ') be in $\mathfrak{D}_{r,a,\delta}$, if S and S' are isomorphic then (S, τ) and (S', τ') are isomorphic. In particular, if τ and τ' are two anti-symplectic involutions on S there exists an automorphism f of S with $f \circ \tau = \tau' \circ f$.*

Proof. Let $M \subset L$ be a sublattice with $(r(M), a(M), \delta(M)) = (r, a, \delta)$, let α_τ and $\alpha_{\tau'}$ be M -markings of (S, τ) and (S', τ') respectively. Let $\eta \in H^0(S, \omega_S) \setminus \{0\}$ and $\eta' \in H^0(S', \omega_{S'}) \setminus \{0\}$. We have the following digram:

$$\begin{array}{ccc} H^2(S, \mathbb{Z}) & & \\ \downarrow \varphi & \searrow \alpha_\tau & \\ & & L \\ & \nearrow \alpha_{\tau'} & \\ H^2(S', \mathbb{Z}) & & \end{array}$$

where φ is a Hodge isometry. Then we have

$$\alpha_\tau(\eta) = (\alpha_\tau \circ \varphi^{-1} \circ \alpha_{\tau'}^{-1})|_{M^\perp}(\alpha_{\tau'}(\eta')).$$

Since $(\alpha_\tau \circ \varphi^{-1} \circ \alpha_{\tau'}^{-1})|_{M^\perp} \in \mathcal{O}(M^\perp)$, we have

$$\overline{\omega}_M(S, \tau) = \overline{\omega}_M(S', \tau').$$

\square

3.3 Applications

Now, we will work with the moduli space $\mathfrak{M}_{8,8,1}$.

Remark: We have $M = I_{1,\tau}(2)$ for the associated Lorentzian sublattice of L , where $I_{p,q}$ stands for the lattice \mathbb{Z}^{p+q} with quadratic form given by the diagonal matrix

$$\text{diag}(\underbrace{1, \dots, 1}_p, \underbrace{-1, \dots, -1}_q)$$

and $\Lambda(d)$ denotes Λ with quadratic form multiplied by d for any lattice Λ and any integer d .

We recall that U is the locus of pairs $(\overline{B}_0, \overline{\Delta}_0)$ in $|\mathcal{O}_{\mathbb{P}^2}(4)| \times |\mathcal{O}_{\mathbb{P}^2}(4)|$ such that \overline{B}_0 and $\overline{\Delta}_0$ are smooth quartics, tangent to each other at eight points lying on a conic. We will denote $\mu_{\overline{B}_0} : X_{\overline{B}_0} \rightarrow \mathbb{P}^2$ the double cover of \mathbb{P}^2 branched over \overline{B}_0 . We define $\mathcal{Q} \subset |\mathcal{O}_{\mathbb{P}^2}(4)| \times |\mathcal{O}_{\mathbb{P}^2}(4)|$ the set of pair $(\overline{B}_0, 2Q)$, where \overline{B}_0 is a smooth quartic and Q a conic such that $\mu_{\overline{B}_0}^*(Q)$ is smooth.

Proposition 3.8. *There is an isomorphism between $U/PGL_3 \cup Q/PGL_3$ and $\mathfrak{M}_{8,8,1}$.*

Proof.

- Step 1: *The map $\mathcal{P} : U/PGL_3 \cup Q/PGL_3 \rightarrow \mathfrak{M}_{8,8,1}$*

We will build $U/PGL_3 \rightarrow \mathfrak{M}_{8,8,1}$; the construction of $Q/PGL_3 \rightarrow \mathfrak{M}_{8,8,1}$ is the same.

First, we have the map $U \rightarrow \mathfrak{M}_{8,8,1}$. Remember that diagram (1) of Section 1 gave a K3 surface S with an involution τ . By page 663 of [16] $(S, \tau) \in \mathfrak{M}_{8,8,1}$. So the diagram (1) gives us the map $U \rightarrow \mathfrak{M}_{8,8,1}$. Now, let $(\overline{B_0}, \overline{\Delta_0})$ and $(\overline{B'_0}, \overline{\Delta'_0})$ be in $|\mathcal{O}_{\mathbb{P}^2}(4)| \times |\mathcal{O}_{\mathbb{P}^2}(4)|$ such that $f(\overline{B_0}, \overline{\Delta_0}) = (\overline{B'_0}, \overline{\Delta'_0})$, where $f \in PGL_3$. We can draw the following commutative diagram.

$$\begin{array}{ccc}
 B_0 & \xlongequal{\quad} & \overline{B_0} \\
 \downarrow & & \downarrow \\
 i \circlearrowleft X & \xrightarrow{\mu} & \mathbb{P}^2 \\
 \downarrow j & & \downarrow f \\
 i' \circlearrowleft X' & \xrightarrow{\mu'} & \mathbb{P}^2 \\
 \downarrow & & \downarrow \\
 B'_0 & \xlongequal{\quad} & \overline{B'_0}
 \end{array}$$

where j is induced by f (the other symbols are the same as in the diagram (1) of Section 1). The map j sends $\mu^{-1}(\overline{\Delta_0}) = \Delta_0 + i(\Delta_0)$ on $\mu'^{-1}(\overline{\Delta'_0}) = \Delta'_0 + i'(\Delta'_0)$. The curves $\overline{\Delta_0}$ and $\overline{\Delta'_0}$ are smooth, so they are irreducible. Then all the curves Δ_0 , $i(\Delta_0)$, Δ'_0 and $i'(\Delta'_0)$ are irreducible. Therefore j sends Δ_0 on Δ'_0 or on $i'(\Delta'_0)$. If j sends Δ_0 on $i'(\Delta'_0)$, we replace j by $i' \circ j$. Now, let $\rho : S \rightarrow X$ and $\rho' : S' \rightarrow X'$ be the double covers branched in Δ_0 and Δ'_0 respectively. We get the following commutative diagram.

$$\begin{array}{ccc}
 \Delta & \xlongequal{\quad} & \Delta_0 \\
 \downarrow & & \downarrow \\
 \tau \circlearrowleft S & \xrightarrow{\rho} & X \\
 \downarrow \varphi & & \downarrow j \\
 \tau' \circlearrowleft S' & \xrightarrow{\rho'} & X' \\
 \downarrow & & \downarrow \\
 \Delta' & \xlongequal{\quad} & \Delta'_0
 \end{array}$$

where φ is induced by j . This implies $(S, \tau) \simeq (S', \tau')$, and we get the map:

$$U/PGL_3 \rightarrow \mathfrak{M}_{8,8,1}.$$

- Step 2: *The inverse function $\mathcal{G} : \mathfrak{M}_{8,8,1} \rightarrow U/PGL_3 \cup Q/PGL_3$*

Let (S, τ) be in $\mathfrak{M}_{8,8,1}$. By [16], $\rho : S \rightarrow X = S/\tau$ is a double cover ramified in a smooth curve of genus 3, Δ and X is a dell Pezzo surface.

Moreover the linear system $| -K_X |$ induces a double cover $\mu : X \rightarrow \mathbb{P}^2$ branched in a smooth quartic of \mathbb{P}^2 , \overline{B}_0 . We have $\rho(\Delta) \in | -2K_X |$, then by Lemma 5.14 of [10], we have $(\overline{B}_0, \mu(\rho(\Delta))) \in U$ or $(\overline{B}_0, \mu(\rho(\Delta))) \in \mathcal{Q}$. Now let (S, τ) and (S', τ') be two isomorphic objects from $\mathfrak{M}_{8,8,1}$. We denote $(\overline{B}_0, \overline{\Delta}_0)$ and $(\overline{B}'_0, \overline{\Delta}'_0)$ the two pairs corresponding to (S, τ) and (S', τ') respectively (here $\overline{\Delta}_0$ and $\overline{\Delta}'_0$ may be a double conic). To have a well defined map from $\mathfrak{M}_{8,8,1}$ to $U/PGL_3 \cup \mathcal{Q}/PGL_3$, we must verify that $(\overline{B}_0, \overline{\Delta}_0)$ and $(\overline{B}'_0, \overline{\Delta}'_0)$ are exchanged by an automorphism of \mathbb{P}^2 . We have an isomorphism $f : S \simeq S'$ with $f \circ \tau = \tau' \circ f$. It induces a commutative diagram

$$\begin{array}{ccccc} S & \xrightarrow{\rho} & X & \longrightarrow & | -K_X | \simeq \mathbb{P}^2 \\ \downarrow f & & \downarrow g & & \downarrow (g^{-1})^* \\ S' & \xrightarrow{\rho'} & X' & \longrightarrow & | -K_{X'} | \simeq \mathbb{P}^2, \end{array}$$

which implies the result.

To finish, we see easily that the composition of \mathcal{G} and \mathcal{P} is the identity. \square

Corollary 3.9. *The involution on U/PGL_3 given by $(\overline{B}_0, \overline{\Delta}_0) \rightarrow (\overline{\Delta}_0, \overline{B}_0)$ induces a rational involution of $\mathfrak{M}_{8,8,1}$ which exchanges the two 2-elementary K3 surfaces $\mathcal{P}(\overline{B}_0, \overline{\Delta}_0) = (S, \tau)$ and $\mathcal{P}(\overline{\Delta}_0, \overline{B}_0) = (\tilde{S}, \tilde{\tau})$. Moreover (S, τ) and $(\tilde{S}, \tilde{\tau})$ are isomorphic if and only if there exists an automorphism f of \mathbb{P}^2 such that $f(\overline{B}_0, \overline{\Delta}_0) = (\overline{\Delta}_0, \overline{B}_0)$.*

We define an open subset of U/PGL_3 by

$$\mathcal{O} = \left\{ (\overline{B}_0, \overline{\Delta}_0) \in U/PGL_3 \mid (\overline{B}_0, \overline{\Delta}_0) \neq (\overline{\Delta}_0, \overline{B}_0) \right\}.$$

Now we are able to answer to the question we asked in the beginning of the section:

Corollary 3.10. *Let $(S, \tau) \in \mathfrak{D}_{8,8,1} \cap \mathcal{P}(\mathcal{O})$. Then S and \tilde{S} are not isomorphic.*

Proof. Since $(S, \tau) \in \mathcal{P}(\mathcal{O})$, (S, τ) and $(\tilde{S}, \tilde{\tau})$ are not isomorphic. Moreover $(S, \tau) \in \mathfrak{D}_{8,8,1}$ therefore by Corollary 3.7, S and \tilde{S} are not isomorphic either. \square

Remark:

- 1) Let $\mathcal{B} := \{ (\overline{\Gamma}, \overline{\Gamma}) \in U/PGL_3 \}$, we have $\mathcal{B} \subset (U/PGL_3) \setminus \mathcal{O}$. Moreover $\dim \mathcal{B} = 6$, via \mathcal{P} , it is the quadruple covers of \mathbb{P}^2 branched over smooth quartics. We also have $\mathcal{B} \subset (U/PGL_3) \setminus (\mathcal{L}/PGL_3)$.
- 2) The quotient variety \mathcal{Q}/PGL_3 has dimension 11.

In fact, we can say even more: S and \tilde{S} are not even derived equivalent. We will denote by $D^b(S)$ the derived category of coherent sheaves on S . Let T_S be the transcendental lattice of S , that is the orthogonal complement to $\text{Pic } S$ in $H^2(S, \mathbb{Z})$. By Theorem 4.2.4. of [19], the categories $D^b(S)$ and $D^b(S')$ are equivalent as triangulated categories if and only if there exists a Hodge isometry between T_S and $T_{S'}$. We have the following theorem.

Theorem 3.11. *Let S and S' be two K3 surfaces such that $D^b(S)$ and $D^b(S')$ are equivalent. If T_S is a 2-elementary sublattice of $H^2(S, \mathbb{Z})$, then S and S' are isomorphic.*

Proof. Let S and S' be two K3 surfaces such that $D^b(S)$ and $D^b(S')$ are equivalent. By Theorem 4.2.4. of [19] we have a Hodge isometry $\rho : T_S \rightarrow T_{S'}$. Let $\alpha : H^2(S, \mathbb{Z}) \cong L$ and $\beta : H^2(S', \mathbb{Z}) \cong L$ be markings of S and S' respectively. The lattices $\alpha(T_S)$ and $\beta(T_{S'})$ are two 2-elementary sublattices of L of signature $(2, x)$. So by Lemma 3.1, $\beta \circ \rho \circ \alpha|_{\alpha(T_S)}^{-1}$ extends to an isometry of L , that we will denote ν . Then $\beta^{-1} \circ \nu \circ \alpha : H^2(S, \mathbb{Z}) \rightarrow H^2(S', \mathbb{Z})$ is a Hodge isometry, therefore by the Global Torelli Theorem for K3 surfaces (see for instance Chapter 10, Theorem 5.3. of [8]), S and S' are isomorphic. \square

Remark: For all 2-elementary K3 surfaces $(S, \tau) \in \mathfrak{D}_{r,a,\delta}$, T_S is a 2-elementary sublattice of $H^2(S, \mathbb{Z})$.

Corollary 3.12. *Let $(S, \tau) \in \mathfrak{D}_{8,8,1} \cap \mathcal{P}(\mathcal{O})$, then $D^b(S)$ and $D^b(\tilde{S})$ are not equivalent.*

Proof. Indeed, if $(S, \tau) \in \mathfrak{D}_{8,8,1}$ then T_S is a 2-elementary lattice. Then if $D^b(S)$ and $D^b(\tilde{S})$ were equivalent, then S and \tilde{S} would be isomorphic, which is false by Corollary 3.10 \square

4 Non-equivalence of dual Relative Compactified Prymians

We will need the following proposition:

Proposition 4.1. *Let S and S' be two complex K3 surfaces. If $S^{[2]}$ and $S'^{[2]}$ are bimeromorphic, then $D^b(S) \sim D^b(S')$.*

Proof. By Lemma 2.6 of [7], if $S^{[2]}$ and $S'^{[2]}$ are birational, there is a Hodge isometry Φ between $H^2(S^{[2]}, \mathbb{Z})$ and $H^2(S'^{[2]}, \mathbb{Z})$, where $H^2(S^{[2]}, \mathbb{Z})$ and $H^2(S'^{[2]}, \mathbb{Z})$ are endowed with the Beauville-Bogomolov form. Moreover, by Section 6 and Lemma 1 Section 9 of [3] we have

$$H^2(S^{[2]}, \mathbb{Z}) = i(H^2(S, \mathbb{Z})) \oplus^\perp \mathbb{Z}\delta_S,$$

where $i : H^2(S, \mathbb{Z}) \rightarrow H^2(S^{[2]}, \mathbb{Z})$ is a Hodge isometry and $\delta_S = \frac{1}{2}\Delta_S$ with Δ_S the class of the diagonal. This implies:

$$\left\{ a \in H^2(S^{[2]}, \mathbb{Z}) \mid B_S(a, i(\eta_S)) \neq 0 \right\} = \{ i(b) \in H^2(S, \mathbb{Z}) \mid b \in T_S \},$$

where $\eta_S \in H^0(S, \omega_S) \setminus \{0\}$, B_S is the Beauville-Bogomolov form of $H^2(S^{[2]}, \mathbb{Z})$ and T_S is the transcendental lattice of S . We have the same results for S' , so Φ induces a Hodge isometry between T_S and $T_{S'}$. Then by Theorem 4.2.4 of [19], S and S' are derived equivalent. \square

We will denote $\mathcal{P}_{(S, \tau)}$ the relative compactified Prymian built from the pair $(S, \tau) \in \mathcal{P}(\mathfrak{L}/PGL_3)$, (see Definition 1.1). If (S, τ) and (S', τ') are two isomorphic 2-elementary K3 surfaces, we have naturally $\mathcal{P}_{(S, \tau)}$ and $\mathcal{P}_{(S', \tau')}$ isomorphic. Now, we can prove the following theorem:

Theorem 4.2. *Let $(S, \tau) \in \mathfrak{D}_{8,8,1} \cap \mathcal{P}(\mathfrak{L}/PGL_3)$ and $(S', \tau') \in \mathcal{P}(\mathfrak{L}/PGL_3)$ such that $\mathcal{P}_{(S, \tau)}$ and $\mathcal{P}_{(S', \tau')}$ are isomorphic, then (S, τ) and (S', τ') are isomorphic.*

Proof. We will denote $M_{(S, \tau)}$ and $M'_{(S, \tau)}$ the varieties defined in Section 1, which are the quotients of $S^{[2]}$ by the involution ι_S and the partial resolution of singularities of $M_{(S, \tau)}$ respectively. We denote $M_{(S', \tau')}$ and $M'_{(S', \tau')}$ the same varieties with (S', τ') instead of (S, τ) . By Theorem 1.6, $M'_{(S, \tau)}$ is bimeromorphic to $\mathcal{P}_{(S, \tau)}$ and $M'_{(S', \tau')}$ is bimeromorphic to $\mathcal{P}_{(S', \tau')}$. Therefore $M'_{(S, \tau)}$ and $M'_{(S', \tau')}$ are bimeromorphic, then $M_{(S, \tau)}$ and $M_{(S', \tau')}$ are bimeromorphic, hence also $M_{(S, \tau)} \setminus \text{Sing } M_{(S, \tau)}$ and $M_{(S', \tau')} \setminus \text{Sing } M_{(S', \tau')}$, so $S^{[2]}$ and $S'^{[2]}$ are bimeromorphic. By Proposition 4.1 we have $D^b(S) \sim D^b(S')$, so by Theorem 3.11, S and S' are isomorphic, and by Corollary 3.7 we have (S, τ) and (S', τ') isomorphic. \square

Corollary 4.3. *The open set $\mathfrak{D}_{8,8,1} \cap \mathcal{P}(\mathfrak{L}/PGL_3)$ of $\mathfrak{M}_{8,8,1}$ provides a 1-to-1 parametrization of the relative compactified Prymians defined in Definition 1.1, and the non trivial rational involution on $\mathfrak{M}_{8,8,1}$ defined in Section 3.3 induces a non trivial involution on the set of the relative compactified Prymians.*

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